SOME EXISTENCE THEOREMS IN THE CALCULUS OF VARIATIONS

III. EXISTENCE THEOREMS FOR NONREGULAR PROBLEMS*

BY

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1. **Outline of the method of proof.** This note is the third of a series, the first and second of which have already appeared in these Transactions.†

In the present paper we shall establish an existence theorem for single-integral problems of the calculus of variations without requiring quasi-regularity. In order to help the reader to find his way through the many details of this proof, we first present a heuristic outline of the method used.

Given an integral $\mathcal{J}(C) = \int F(z, \dot{z})dt$, where $z = (z^1, \cdots, z^q)$, we seek to find a rectifiable curve (in q-dimensional space) joining two fixed points z_1, z_2 for which this integral is least. It is always possible to find a minimizing sequence $\{C_n\}$; that is, a sequence for which $\mathcal{J}(C_n)$ tends to the greatest lower bound μ of $\mathcal{J}(C)$ on the class of curves under consideration. Moreover, it is easy to find hypotheses on $\mathcal{J}(C)$ which will ensure that a minimizing sequence exist which has a curve of accumulation C_0. That is, the curves C_n have representations $z = z_n(t)$, $(0 \leqq t \leqq 1)$, for which $z_n(t)$ tends uniformly to a limit function $z_0(t)$. If now we have some method of selecting the C_n so that $z_n'(t)$ tends almost everywhere to $z_0'(t)$, then $\mathcal{J}(C_n) \to \mathcal{J}(C_0)$, and the curve C_0 is the one sought.

Suppose that a curve C: $z = z(t)$, $(t_1 \leqq t \leqq t_2)$, in q-dimensional space minimizes an integral $\mathcal{J}(C)$ in the class of all curves of class‡ D' joining two fixed points z_1 and z_2. As is well known, the equation

$$(1.1) \qquad F_i(z(t), z'(t)) = \int_{t_1}^{t} F_{z^i}(z(t), z'(t)) + c_i$$

holds. Also, at each corner $z(t_0)$ of C the relations

$$(1.2) \qquad F_i(z(t_0), z'(t_0 - 0)) = F_i(z(t_0), z'(t_0 + 0))$$

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† These Transactions, vol. 44 (1938), pp. 429–438; pp. 439–453.

‡ A curve C is of class D' if it has a representation $z = z(t)$, $(t_1 \leqq t \leqq t_2)$, such that the functions $z^i(t)$ are continuous and the interval $[t_1, t_2]$ contains a finite number of points between which $z'(t)$ is non-vanishing and uniformly continuous. (At these points $z'(t)$ may be $(0, \cdots, 0)$ or may fail to exist.)

and*

$$\Omega(z(t_0),\ z'(t_0 - 0),\ z'(t_0 + 0)) \leqq 0 \tag{1.3}$$

hold, where

$$\Omega(z,\ p,\ r) = p^\alpha F_{z^\alpha}(z,\ r) - r^\alpha F_{z^\alpha}(z,\ p). \tag{1.4}$$

If now we choose a sequence of polygons Π_n joining z_1 to z_2 and such that $\mathcal{J}(\Pi_n)$ tends to the lower bound μ of $\mathcal{J}(C)$ on the class of curves under consideration, we can not immediately make any statement similar to (1.1) or (1.3) for Π_n. However, if the number of vertices of Π_n is s_n, and Π_n actually minimizes $\mathcal{J}(C)$ in the class of all polygons joining z_1 and z_2 and having not more than s_n vertices, then it is to be expected that relations (1.1), (1.2), and (1.3) hold in an approximate sense for Π_n.

We now define an *approach set at z* to be an aggregate of vectors p such that

$$F_i(z,\ p_1) - F_i(z,\ p_2) = 0, \qquad i = 1, \cdots, q, \tag{1.5}$$

for each pair p_1, p_2 of vectors of the set. In this terminology, equation (1.2) states that $z'(t_0-0)$ and $z'(t_0+0)$ belong to an approach set at $z(t_0)$. Suppose then that the polygons Π_n have been chosen as above so that Π_n minimizes $\mathcal{J}(C)$ in the class of polygons which have not more than s_n vertices and which join z_1 to z_2. We suppose that these are represented by functions $z=z_n(t)$, $(0\leqq t\leqq 1)$, which are piecewise linear on $[0, 1]$. Also we suppose that we have already chosen a convergent sequence, so that $z_n(t)$ tends to a limit function $z_0(t)$ uniformly on $[0, 1]$. If we fix on any number t_0 in $[0, 1]$, it is possible to choose a subsequence $\{\Pi_m\}$ of the sequence $\{\Pi_n\}$ in such a way that the vectors $z_m'(t_0+0)$ tend to a limit p_0. Whenever m is large and t is near t_0, the point $z_m(t)$ is near $z_0(t_0)$ and the vector $z_m(t_0+0)$ is near p_0. If now equation (1.1) holds, at least approximately, for Π_m, the value of $F_i(z_m(t),\ z_m'(t))$ can change only a small amount on a short arc of Π_m. For by (1.1) the change in $F_i(z_m(t),\ z_m'(t))$ is equal to the integral of F_{z^i} over a short interval. Hence for all t near t_0 the values of the $F_i(z_m(t),\ z_m'(t))$ are nearly equal to the values of $F_i(z_m(t_0),\ z_m'(t_0))$, and these in turn are nearly equal to $F_i(z_0(t_0),\ p)$. Thus for all large m and for all t near t_0 the equations

$$F_i(z_0(t_0),\ z_m'(t)) = F_i(z_0(t_0),\ p_0), \qquad i = 0, 1, \cdots, q,$$

are almost true, and we may expect that there is an approach set A such that $z_m'(t)$ is in or near A whenever m is large and $t-t_0$ small.

Suppose now that each such approach set contains only a finite number

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* Cf. I, Theorem 1.

of unit vectors, which can be set in an order $p_1, p_2, \cdots, p_k$ such that

$$\Omega(z_0(t_0), p_j, p_i) > 0, \qquad j > i. \tag{1.6}$$

Now (1.3) must hold approximately on Π_m, and for t near t_0 and m large the directions $z_m'(t)/|z_m'(t)|$ are each near some p_j; hence by (1.3) and (1.6) we find that a side with direction near p_j cannot precede one with direction near p_i if $j>i$. Thus the arc of Π_m near $z_0(t_0)$ can be split into subarcs, the first of which consists of sides with directions near p_1, the second of sides with directions near p_2, and so on. That is, each such arc is almost a line segment. So is its limit arc; and for line segments l_n tending to a limit l_0 it is clear that $\mathcal{J}(l_n)\to\mathcal{J}(l_0)$. Thus our point t_0 is in an interval along which the integrand $F(z_n, \dot{z}_n)$ converges, with arbitrarily small error, to $F(z_0, \dot{z}_0)$.

This argument, applied to all t_0, would yield $\mathcal{J}(\Pi_m)\to\mathcal{J}(C_0)$. Since $\mathcal{J}(\Pi_m)\to\mu$, we have $\mathcal{J}(C_0)=\mu$, and C_0 is the curve sought.

In the following pages the argument just suggested will be generalized and made rigorous.

2. **Choice of a minimizing sequence.** We now suppose that we are given an integral $\mathcal{J}(C)$ in parametric form, and seek the minimum of $\mathcal{J}(C)$ in the class of all rectifiable curves C joining two given points z_1 and z_2. We suppose that $\mathcal{J}(C)$ satisfies the following condition:

(2.1) *For every constant M there is a number L_M such that all curves C joining z_1 and z_2 and giving $\mathcal{J}(C)$ a value less than or equal to M have lengths not greater than L_M.*

For example, (2.1) is satisfied if there is a number $c>0$ such that

$$F(z, z') \geqq c\,|z'|/(1+|z|).$$

We shall reserve the word "vertex" for points on a polygon at which successive sides join; that is, the initial and final points will not be called vertices.

LEMMA 1. *Let $\mathcal{J}(C)$ satisfy* (2.1). *Let K_s be the class of all polygons joining a point z_1 to a point z_2 and having not more than s vertices. Then for every s the class K_s contains a polygon which minimizes $\mathcal{J}(C)$ on the class K_s.*

Let μ_s be the greatest lower bound of $\mathcal{J}(C)$ on the class K_s. We establish a correspondence between the polygons of K_s and the points $(\zeta^1, \cdots, \zeta^{sq})$ of sq-dimensional space by writing the coordinates of the first vertex, then those of the second, and so on.* By (2.1), there is a number R such that if any coordinate of the point ζ representing a polygon Π exceeds R in absolute value, then $\mathcal{J}(\Pi)>\mu+1$. So the minimum of $\mathcal{J}(\Pi)$ on the class K_s is to be

* We may suppose that every polygon of K_s has s vertices, because if it has less we can insert points of division in the sides and increase the number of vertices up to s.

sought among the images of the set $|\zeta| \leqq Rsq$. This is bounded and closed, and on it $\mathcal{J}(\Pi)$ is continuous; hence the minimum is assumed.

Now we introduce the notation:

(2.2) *K is the class of all rectifiable curves joining the two distinct fixed points z_1 and z_2, and μ is the greatest lower bound of $\mathcal{J}(C)$ on the class K.*

Then we have the following lemma:

LEMMA 2. *If $\mathcal{J}(C)$ satisfies* (2.1), *there exists a sequence of polygons Π_n: $z=z_n(t)$, $(0\leqq t\leqq 1)$, in the class K with the properties*:

(a) $\mathcal{J}(\Pi_n)\rightarrow\mu$.

(b) *s_n being the number of vertices of Π_n, and K_{s_n} being the subclass of K consisting of polygons of not more than s_n vertices, Π_n minimizes $\mathcal{J}(C)$ on the class K_{s_n}.*

(c) *There is a constant L such that $|\dot{z}_n(t)|\leqq L$ for all n and all t in* $[0, 1]$.

(d) *$z_n(t)$ converges uniformly to a limit function $z_0(t)$,* $(0\leqq t\leqq 1)$.

Let $\{C_n\}$ be a sequence of curves of K for which $\mathcal{J}(C_n)\rightarrow\mu$. For each fixed n we inscribe in C_n a sequence of polygons $\{\Pi_k^*\}$ with sides tending to 0. It is well known that under these conditions $\mathcal{J}(\Pi_k^*)$ tends to $\mathcal{J}(C_n)$. Hence we can find a particular k for which $\mathcal{J}(\Pi_k^*)<\mathcal{J}(C_n)+1/n$.

Let s_k^* be the number of vertices of Π_k^*. By Lemma 1, there is a polygon Π_n of s_n vertices, where $s_n\leqq s_n^*$, which minimizes $\mathcal{J}(C)$ on the class $K_{s_n^*}$. Therefore $\mathcal{J}(\Pi_n)\leqq\mathcal{J}(\Pi_n^*)<\mathcal{J}(C)+1/n$. Since Π_n minimizes $\mathcal{J}(C)$ in the class $K_{s_n^*}$ and is in the class $K_{s_n}\subset K_{s_n^*}$, it is clear that Π_n minimizes $\mathcal{J}(C)$ on K_{s_n}. Also,

$$\limsup_{n\rightarrow\infty}\mathcal{J}(\Pi_n)\leqq\lim_{n\rightarrow\infty}\mathcal{J}(C_n)=\mu.$$

But Π_n is in K; so $\mathcal{J}(\Pi_n)\geqq\mu$, and $\liminf_{n\rightarrow\infty}\mathcal{J}(\Pi_n)\geqq\mu$. Thus (a) and (b) are established.

There is no loss of generality in supposing $\mathcal{J}(\Pi_n)<\mu+1$ for all n. Hence, by hypothesis (2.1), there is an L such that each Π_n has length $\mathcal{L}(\Pi_n)$ less than L. On Π_n we introduce the parameter $t=s/\mathcal{L}(\Pi_n)$, where s is arc length. Then Π_n has a representation $z=z_n(t)$, $(0\leqq t\leqq 1)$, with $|\dot{z}_n(t)|\leqq\mathcal{L}(C_n)<L$, and (c) is satisfied.

Finally, the $z_n(t)$ all satisfy a Lipschitz condition with constant L; so by Ascoli's theorem there is a subsequence (which we may without loss of generality suppose to be the whole sequence) for which $z_n(t)$ converges uniformly to a limit function $z_0(t)$. This completes the proof of the lemma.

Remark. If a sequence $\{\Pi_n\}$ satisfies (a), (b), (c), and (d) of Lemma 2, so does every subsequence of $\{\Pi_n\}$.

Before stating the next lemma we introduce a definition.

(2.3) *If* $[\alpha, \beta]$ *is an interval having points in common with* $[0, 1]$ *and* $t_{n,1}, t_{n,2}, \cdots, t_{n,k}$ *are (in that order) the values of* t *in* $[\alpha, \beta]$ *which define vertices of* Π_n, *then*

$$\theta(n, \alpha, \beta) \equiv t_{n,1}, \qquad \Theta(n, \alpha, \beta) \equiv t_{n,k}.$$

If no value of t *in* $[\alpha, \beta]$ *defines a vertex of* Π_n, *we define*

$$\theta(n, \alpha, \beta) = \Theta(n, \alpha, \beta) = (\alpha + \beta)/2. \tag{2.4}$$

Then we have the following lemma:

LEMMA 3. *If* $\{\Pi_n\}$ *is the sequence of Lemma* 2, *then for every* $\epsilon > 0$ *and every* t_0 *in* $[0, 1]$ *there is a* $\delta > 0$ *such that*

$$| F_i(z_0(t_0), z_n'(t')) - F_i(z_0(t_0), z_n'(t'')) | \leqq \epsilon, \qquad i = 1, \cdots, q, \tag{2.5}$$

if t_0, t', *and* t'' *are all in the (open) interval* $(\theta(n, t_0-\delta, t_0+\delta), \Theta(n, t_0-\delta, t_0+\delta))$ *and* $z_n(t')$ *and* $z_n(t'')$ *are not vertices*† *of* Π_n.

We shall suppose, to be specific, that $t' < t''$. Let $t_1, t_2, \cdots, t_k$ define successive vertices of Π_n such that $t_1 < t' < t_2$ and $t_{k-1} < t'' < t_k$. By the definition of θ and Θ we know that

$$t_0 - \delta < t_1 < t_k < t_0 + \delta. \tag{2.6}$$

From Π_n we now form the polygon $\Pi_n^j(\tau)$ by displacing each of the vertices $z(t_2), \cdots, z(t_{k-1})$ by the amount $(0, \cdots, \tau, 0, \cdots, 0)$, the τ being in the jth place. Because of the minimizing property of Π_n we have

$$\frac{d}{d\tau} \mathcal{J}(\Pi_n^j(\tau)) \Big|_{\tau=0} = 0. \tag{2.7}$$

But if we apply formula (2.8) of I to the successive sides of $\Pi_n^j(\tau)$ defined by $t_1 \leqq t \leqq t_2, \cdots, t_{k-1} \leqq t \leqq t_k$, and then add, we obtain

$$\begin{aligned} \frac{d}{d\tau} \mathcal{J}(\Pi_n^j(\tau)) \Big|_{\tau=0} &= F_j(z_n(\bar{t}), z_n'(\bar{t})) - F_j(z_n(\bar{t}), z_n'(\bar{t})) \\ &\quad + \int_{t_1}^{t_2} F_{zj}(z_n, \dot{z}_n) \frac{t - t_1}{t_2 - t_1} dt + \int_{t_2}^{t_{k-1}} F_{zj}(z_n, \dot{z}_n) dt \\ &\quad + \int_{t_{k-1}}^{t_k} F_{zj}(z_n, \dot{z}_n) \frac{t_k - t}{t_k - t_{k-1}} dt, \end{aligned} \tag{2.8}$$

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† It is easy to see that this restriction is essentially no restriction, for if t' defines a vertex, we first write (2.5) with t' replaced by $t'+h$, $h>0$, and then let $h\to 0$. For small h the value of $z'(t'+h)$ is constantly equal to $z'(t'+0)$; whence we find (2.5) with t' replaced by $t'+0$. Similarly, we could replace t' by $t'-0$, and analogously for t''.

where $t_1 < \bar{t} < t_2$ and $t_{k-1} < \tilde{t} < t_k$. All the curves Π_n lie in a bounded closed subset of the space, and all $|\dot{z}_n|$ are less than L. Hence the functions $F_{zj}(z_n, \dot{z}_n)$ are bounded, say less than N in absolute value. Then the sum of the three integrals on the right is at most $N(t_k - t_1)$ in absolute value, so that

$$0 = F_j(z_n(\bar{t}), z_n'(\bar{t})) - F_j(z_n(\tilde{t}), z_n'(\tilde{t})) + \theta_j, \tag{2.9}$$

where $|\theta_j| \leqq N(t_k - t_{k-1}) \leqq 2N\delta$.

On the interval (t_1, t_2) the derivative $z_n'(t)$ is constantly equal to $z_n'(t')$, and on (t_{k-1}, t_k) it is constantly equal to $z_n'(t'')$. Hence from (2.9) we obtain

$$\begin{aligned} &| F_j(z_n(t'), z_n'(t')) - F_j(z_n(t''), z_n'(t''))| \\ &\qquad \leqq | F_j(z_n(t'), z_n'(t')) - F_j(z_n(\bar{t}), z_n'(t'))| \\ &\qquad\quad + | F_j(z_n(t''), z_n'(t'')) - F_j(z_n(\tilde{t}), z_n'(t''))| + |\theta_j|. \end{aligned} \tag{2.10}$$

Because of the continuity of F_j, each of the first two terms on the right is less than $\epsilon/3$ if $|z_n(\bar{t}) - z_n(t')|$ and $|z_n(\tilde{t}) - z_n(t'')|$ are less than a certain $\gamma > 0$. This is surely the case if $|\bar{t} - t'|$ and $|\tilde{t} - t''|$ are less than γ/L, which again is surely true if $2\delta < \gamma/L$. Also, if $\delta < \epsilon/6N$, the term $|\theta_j|$ is at most $2N\delta < \epsilon/3$. Hence, if δ is any number less than the smaller of $\gamma/2L$ and $\epsilon/6N$, inequality (2.10) implies (2.5).

Remark. From the last sentence of the proof it is obvious that δ can be chosen as near to 0 as desired.

3. **Directions of the sides of the minimizing sequence.** The next lemma will explain the name "approach set" given to the sets satisfying (1.5).

LEMMA 4. *Let $\{\Pi_n\}$ be a sequence of polygons satisfying* (a), (b), (c), *and* (d) *of Lemma* 2, *and let $z_1 \neq z_2$. Let t_0 be interior to* (0, 1). *Then there is a subsequence $\{\Pi_m\}$ of $\{\Pi_n\}$ and an approach set A at $z_0(t_0)$ such that for every $\epsilon > 0$ there is a $\delta > 0$ and an m_0 such that $z_m'(t)$ is in the ϵ-neighborhood of A whenever $z_m(t)$ is not a vertex, $m > m_0$, and*

$$\theta(m, t_0 - \delta, t_0 + \delta) < t < \Theta(m, t_0 - \delta, t_0 + \delta). \tag{3.1}$$

If there is a $\delta > 0$ and a subsequence $\{\Pi_m\}$ such that not more than one t in $(t_0 - \delta, t_0 + \delta)$ defines a vertex of Π_m, then the conclusion holds vacuously, for then $\theta(m, t_0 - \delta, t_0 + \delta) = \Theta(m, t_0 - \delta, t_0 + \delta)$, and (3.1) is not satisfied by any t.

Otherwise, for each integer m there are infinitely many n_m such that the interval

$$(\theta(n_m, t_0 - m^{-1}, t_0 + m^{-1}), \ \Theta(n_m, t_0 - m^{-1}, t_0 + m^{-1})) \tag{3.2}$$

has length greater than zero. We can suppose n_m so chosen that $n_1 < n_2 < \cdots$.

In each interval (3.2) we choose a number t_{n_m} which does not define a vertex of Π_{n_m}. The vectors $z'_{n_m}(t_{n_m})$ all have lengths less than or equal to L; hence they have an accumulation vector p_0. We select a subsequence $\{m\}$ of the sequence $\{n_m\}$ such that $z'_m(t_m)\to p_0$.

Now let A be the approach set at $z_0(t_0)$ which contains p_0, and let U be the set of all vectors u whose distance from A is less than ϵ. Then U is clearly open. We are to show that $z'_m(t)$ is in U if t and m satisfy the conditions of our lemma. If V is the set of all vectors v of length $|v|\leqq L$ which are not in U, then V is bounded and closed and has no point in common with A. Hence the function

$$\sum_{i=1}^{q}\left|F_i(z_0(t_0),\,p_0)-F_i(z_0(t_0),\,v)\right| \tag{3.3}$$

is positive for all v in V. Being continuous† on V, it is greater than a positive number 3γ.

Now

$$\begin{aligned}\sum_{i=1}^{q}&\left|F_i(z_0(t_0),\,z'_m(t))-F_i(z_0(t_0),\,p_0)\right|\\ &\leqq\sum_{i=1}^{q}\left|F_i(z_0(t_0),\,z'_m(t))-F_i(z_m(t),\,z'_m(t))\right|\\ &\quad+\sum_{i=1}^{q}\left|F_i(z_m(t),\,z'_m(t))-F_i(z_m(t_m),\,z'_m(t_m))\right|\\ &\quad+\sum_{i=1}^{q}\left|F_i(z_m(t_m),\,z'_m(t_m))-F_i(z_0(t_0),\,p_0)\right|.\end{aligned} \tag{3.4}$$

Here, as in Lemma 3, the parameter t is $s/\mathcal{L}(\Pi_m)$, so that $|z'_m(t)|=\mathcal{L}(\Pi_m)\geqq|z_2-z_1|>0$. Thus the arguments in the first term on the right have a bounded closed range and $z'_m(t)$ is bounded from 0. Hence the first term is less than γ if $|z_0(t_0)-z_m(t)|$ is smaller than a certain number κ. Since $z_m(t)$ tends uniformly to $z_0(t)$ and $z_0(t)$ is continuous, this is true if δ is small enough and m is greater than a certain m_0.

In the last term on the right, $z'_m(t_m)$ tends to p_0 and $z_m(t_m)$ to $z_0(t_0)$; so this term is less than γ if m is greater than a certain m_2.

If δ is small enough and m is greater than a certain m_3, then by Lemma 3 the third term is less than γ whenever t does not define a vertex of Π_m and (3.1) holds. Hence if δ is small enough and $m>m_0\equiv\max(m_1,\,m_2,\,m_3)$, then for all t satisfying the requirements of our lemma we have

† The only point at which $F_i(z_0(t_0),\,v)$ is discontinuous is $v=0$, which is a limit point of A and therefore not in V.

$$\sum_{i=1}^{q} \left| F_i(z_0(t_0), z_m'(t)) - F_i(z_0(t_0), p_0) \right| < 3\gamma.$$

But if $z_m'(t)$ were in V, this expression (see (3.3)) would exceed 3γ. So $z_m'(t)$ is not in V. Also, it satisfies the condition $|z_m'(t)| \leqq L$. Hence it must belong to U, and the lemma is established.

Remark. Again it is clear that the δ can be chosen as small as desired.

4. **Proof of the principal theorem.** We now impose a new hypothesis on the integral $\mathcal{J}(C)$. We shall make the following supposition:

(4.1) *For every z, each approach set A at z is the sum of a finite number of convex sets $A_1, \cdots, A_k$, which can be so ordered that*

$$\Omega(z, p_i, p_j) < 0$$

if p_i is in A_i and p_j in A_j, $(i<j)$.

Our principal theorem is the following:

THEOREM 1. *If $F(z, z')$ satisfies conditions* (2.1) *and* (4.1), *then in the class K of all rectifiable curves joining two distinct points z_1 and z_2 there is a curve which minimizes the integral*

$$\mathcal{J}(C) = \int_C F(z, \dot{z})dt.$$

We select a sequence of polygons Π_n and a curve C_0 satisfying the conclusions of Lemma 2, and we define

$$\phi_n(t) = \int_0^t F(z_n, \dot{z}_n)dt, \qquad 0 \leqq t \leqq 1;\ n = 0, 1, 2, \cdots. \tag{4.2}$$

Since $|\dot{z}_n(t)| \leqq L$, the integrands are bounded, and the ϕ_n all satisfy the same Lipschitz condition. Hence by Ascoli's theorem we can select a subsequence (we suppose it to be the whole sequence) such that $\phi_n(t)$ tends uniformly to a limit function $\phi(t)$. By (a) of Lemma 2,

$$\phi(1) = \lim_{n\to\infty} \phi_n(1) = \lim_{n\to\infty} \mathcal{J}(\Pi_n) = \mu. \tag{4.3}$$

Since

$$\phi_0(1) = \mathcal{J}(C_0), \tag{4.4}$$

we must show then that

$$\phi_0(1) = \phi(1) = \mu. \tag{4.5}$$

We shall in fact show that

$$\phi_0(t) = \phi(t), \qquad 0 \leqq t \leqq 1. \tag{4.6}$$

Let t_0 be a point interior to (0, 1) at which $\phi'(t)$ and $\phi_0'(t)$ are both defined. We shall first show

$$\phi'(t_0) = \phi_0'(t_0). \tag{4.7}$$

This will be established if we can prove the following statement:

(4.8) *For every positive number γ there is a $t^* > t_0$ such that if $t_0 < \rho < \sigma < t^*$, then*

$$\left| [\phi(\sigma) - \phi(\rho)] - [\phi_0(\sigma) - \phi_0(\rho)] \right| \leqq \gamma(\sigma - \rho). \tag{4.9}$$

For if (4.8) holds, then by continuity (4.9) holds with ρ replaced by t_0. Dividing by $\sigma - t_0$ and letting σ tend to t_0 we obtain

$$\left| \phi'(t_0) - \phi_0'(t_0) \right| \leqq \gamma.$$

Since γ is an arbitrary positive number, this can be true only if (4.7) holds. We therefore take γ to be a positive number and proceed to establish (4.8).

Let us first dispose of the relatively simple case in which there is a subsequence $\{\Pi_h\}$ and a $\delta > 0$ such that either (a) $\lim_{h\to\infty} \theta(h, t_0 - \delta, t_0 + \delta) > t_0$ or (b) $\lim_{h\to\infty} \Theta(h, t_0 - \delta, t_0 + \delta) \leqq t_0$. If (a) holds, we denote $\lim \theta$ by t^*; if (b) holds, we let t^* equal $t_0 + \delta$. Let ρ, σ be numbers such that $t_0 < \rho < \sigma < t^*$. Then if h is large enough, the interval $[\rho, \sigma]$ is contained in case (a) in $[t_0 - \delta, \theta(h, t_0 - \delta, t_0 + \delta)]$, and in case (b) in $[\Theta(h, t_0 - \delta, t_0 + \delta), t_0 + \delta]$. In either case, $z_h(t)$ is linear on $[\rho, \sigma]$ for all large h. The same is then true of its limit $z_0(t)$, and it follows at once that $z_h'(t)$ tends to $z_0'(t)$ uniformly on $[\rho, \sigma]$. Therefore

$$\phi(\sigma) - \phi(\rho) = \lim_{h\to\infty} \{\phi_h(\sigma) - \phi_h(\rho)\} = \lim_{h\to\infty} \int_\rho^\sigma F(z_h, z_h')dt$$

$$= \int_\rho^\sigma F(z_0, z_0')dt = \phi_0(\sigma) - \phi_0(\rho);$$

and (4.8) is established for all positive γ at once.

Now we return to the general case. By Lemma 4, there is an approach set A at $z_0(t_0)$ with the properties there specified. If p_0 is any fixed vector in A, then for every vector p in A the equations

$$F_i(z_0(t_0), p) = F_i(z_0(t_0), p_0), \qquad i = 1, \cdots, q,$$

hold, by definition. If we denote the right-hand members of these equations by $l_1, \cdots, l_q$, respectively, then on multiplying by p^i and summing we find, for all p in A, that

$$F(z_0(t_0), p) = l_\alpha p^\alpha.$$

The set of vectors p in A with $|p| \leqq L$ is bounded and closed, and the function F is continuous. Therefore there is a $\lambda > 0$ such that

$$(4.10) \qquad |F(z, p) - l_\alpha p^\alpha| < \gamma/2$$

if $|z - z_0(t_0)| < \lambda$ and p is in the λ-neighborhood U_λ of the intersection of A with the sphere $|p| \leqq L$.

The set A is the sum of convex sets $A_1, \cdots, A_k$. Let us denote by $U_{j,\epsilon}$ the ϵ-neighborhood of the intersection of A_j with the sphere $|p| \leqq L$. Then $U_\epsilon = U_{1,\epsilon} + \cdots + U_{k,\epsilon}$. By hypothesis, the A_j have the property that

$$(4.11) \qquad \Omega(z_0(t_0), p_i, p_j) < 0$$

if p_i is in A_i, p_j is in A_j, and $i < j$. Therefore on the bounded closed set of arguments (z, p_i, p_j) which satisfy the conditions $z = z_0(t_0)$, p_i in A_i, p_j in A_j, $(i<j)$, $|z_2 - z_1| \leqq |p_i| \leqq L$, $|z_2 - z_1| \leqq |p_j| \leqq L$ this function has a negative upper bound -2κ. Since it is continuous, we find that there is an $\eta > 0$ such that

$$(4.12) \qquad \Omega(z, p_i, p_j) < -\kappa$$

whenever p_i is in $U_{i,\eta}$, p_j is in $U_{j,\eta}$, $(i<j)$, $|z_2 - z_1| \leqq |p_i| \leqq L$, $|z_2 - z_1| \leqq |p_j| \leqq L$, and $|z - z_0(t_0)| < 3\eta$.

Now define

$$(4.13) \qquad \epsilon = \min(\eta, \lambda).$$

We use this for the ϵ in Lemma 4, and choose (see the remark after the lemma) a value for δ which is less than $\epsilon/2(L+1)$. Furthermore, from the sequence $\{\Pi_m\}$ of that lemma we discard those polygons (finite in number) for which the inequality

$$(4.14) \qquad |z_m(t) - z_0(t)| < \epsilon/2, \qquad 0 \leqq t \leqq 1,$$

fails to hold. Then, by Lemma 4, together with (4.10) and (4.13), we find:

(4.15) *For all functions $z_m(t)$ and all t such that*

$$\theta(m, t_0 - \delta, t_0 + \delta) < t < \Theta(m, t_0 - \delta, t_0 + \delta),$$

the inequality

$$|F(z_m(t), \dot{z}_m(t)) - l_\alpha \dot{z}_m^\alpha(t)| < \gamma/2$$

holds, and $z_m'(t)$ is in one of the sets $U_{1,\epsilon}, \cdots, U_{k,\epsilon}$ if $z_m'(t)$ is defined.

Let t_1, t_2, t_3 define three consecutive vertices of Π_m, and let the inequalities

$$\theta(m, t_0 - \delta, t_0 + \delta) \leqq t_1 < t_2 < t_3 \leqq \Theta(m, t_0 - \delta, t_0 + \delta)$$

hold. The sides of Π_m corresponding to $[t_1, t_2]$ and $[t_2, t_3]$ have directions

$z_m'(t_1+0)$, $z_m'(t_2+0)$. By (4.15) these belong respectively to neighborhoods $U_{i,\epsilon}$, $U_{j,\epsilon}$; and the inequalities

$$|z_2 - z_1| \leqq |z_m'(t)| \leqq L$$

always hold. We now show that i is less than j. Suppose the contrary. If we interchange these two sides of Π_m, we obtain a new polygon Π_m^*, and by formula (3.19) of I

$$(4.16)\quad \begin{aligned} \mathcal{J}(\Pi_m^*) - \mathcal{J}(\Pi_m) = -|z_m(t_3) - z_m(t_2)|\,|z_m(t_2) - z_m(t_1)| \\ \cdot\Omega(\bar{z}, z_m'(t_1+0), z_m'(t_2+0)), \end{aligned}$$

where $\bar{z}$ is in the parallelogram determined by the two sides. However, $|t_1-t_0|<\delta<\epsilon/2L$; so $|z_m(t_1)-z_m(t_0)|<L\cdot\epsilon/2L=\epsilon/2$, while by (4.14), $|z_m(t_0)-z_0(t_0)|<\epsilon/2$. Hence $|z_m(t_1)-z_0(t_0)|<\epsilon$. The sides of this parallelogram have lengths less than $|t_3-t_1|L<2\delta L<\epsilon$. Thus the whole parallelogram lies in the 3ϵ-neighborhood of $z_0(t_0)$, and in particular $|\bar{z}-z_0(t_0)|<3\epsilon\leqq 3\eta$, by (4.13). Therefore by (4.12) and (4.13) the factor Ω in (4.16) is positive,† and $\mathcal{J}(\Pi_m^*)<\mathcal{J}(\Pi_m)$. This contradicts the minimizing property of Π_m, and establishes the inequality $i<j$.

We have therefore established, as a sort of addition to (4.15), that for t between θ and Θ no side of Π_m whose direction is in a $U_{j,\epsilon}$ is followed by one whose direction is in a $U_{i,\epsilon}$ with $i<j$. Let us now define

$$(4.17)\qquad t_{m,0} = \theta(m, t_0-\delta, t_0+\delta), \qquad t_{m,k} = \Theta(m, t_0-\delta, t_0+\delta).$$

By the previous remark, there are numbers $t_{m,i}$ defining vertices of Π_m such that

$$t_{m,0} \leqq t_{m,1} \leqq \cdots \leqq t_{m,k},$$

and if $t_{m,j-1}<t<t_{m,j}$ and $z_m'(t)$ is defined, then $z_m'(t)$ is in $U_{j,\epsilon}$. We now select a subsequence $\{\Pi_h\}$ of $\{\Pi_m\}$ such that for each $j=0, 1, \cdots, k$ the limit

$$\tau_j = \lim_{h\to\infty} t_{h,j}$$

exists. We may suppose $\tau_0\leqq t_0<\tau_k$, for otherwise we are back to the simpler case first considered.

There is a first j such that $\tau_j>t_0$; hence $\tau_{j-1}\leqq t_0<\tau_j$. Let ρ, σ be numbers such that $t_0<\rho<\sigma<\tau_j$. For all large h the inequalities $\tau_{h,j-1}<\rho<\sigma<\tau_{h,j}$ hold, so that $\dot{z}_h(t)$ is in $U_{j,\epsilon}$ if $\rho\leqq t\leqq\sigma$. The set $U_{j,\epsilon}$ is the ϵ-neighborhood of the

† Recall that the interchange of p_i and p_j in (4.12) changes the sign of Ω.

intersection of two convex sets, A_j and the sphere $|p| \leqq L$; so $U_{j,\epsilon}$ is convex. By Jensen's inequality (in geometric form), if $\rho \leqq t < t' \leqq \sigma$, the vector

$$\frac{1}{t'-t}\int_t^{t'} \dot{z}_h(t)dt = \frac{z_h(t') - z_h(t)}{t'-t}$$

is in the closure $\overline{U}_{j,\epsilon}$. Letting $h \to \infty$, we find that the limit

$$\frac{z_0(t') - z_0(t)}{t'-t}$$

is also in $\overline{U}_{j,\epsilon}$. Now let $t' \to t$; we find that the vector $z_0'(t)$ is in $\overline{U}_{j,\epsilon}$ if it is defined. Since the zero vector is in $U_{j,\epsilon}$, we see that in any case $\dot{z}_0(t)$ is in $\overline{U}_{j,\epsilon}$ for $\rho \leqq t < \sigma$.

Turning now to (4.10), we therefore obtain

$$|F(z_0(t), \dot{z}_0(t)) - l_\alpha \dot{z}_0^\alpha(t)| \leqq \gamma/2$$

if $\rho \leqq t < \sigma$. Hence

$$\begin{aligned} |\phi_0(\sigma) - \phi_0(\rho) - l_\alpha\{z_0^\alpha(\sigma) - z_0^\alpha(\rho)\}| &= \left|\int_\rho^\sigma \{F(z_0, \dot{z}_0) - l_\alpha \dot{z}_0^\alpha\} dt\right| \\ &\leqq \gamma(\sigma - \rho)/2. \end{aligned} \tag{4.18}$$

On the other hand, by (4.15)

$$\begin{aligned} |\phi_h(\sigma) - \phi_h(\rho) - l_\alpha\{z_h^\alpha(\sigma) - z_h^\alpha(\rho)\}| &= \left|\int_\rho^\sigma \{F(z_h, \dot{z}_h) - l_\alpha \dot{z}_h^\alpha\} dt\right| \\ &< \gamma(\sigma - \rho)/2. \end{aligned} \tag{4.19}$$

If we here let $h \to \infty$, we obtain

$$|\phi(\sigma) - \phi(\rho) - l_\alpha\{z_0^\alpha(\sigma) - z_0^\alpha(\rho)\}| \leqq \gamma(\sigma - \rho)/2. \tag{4.20}$$

From (4.18) and (4.20) we at once obtain (4.9). Thus statement (4.8), and with it (4.7), is established.

Since $\phi(0) = \phi_0(0) = 0$ and (4.7) holds for almost all t, and since the functions ϕ and ϕ_0 are both absolutely continuous, it follows that $\phi(t) \equiv \phi_0(t)$, so that (4.5) is established. Therefore the curve $z = z_0(t)$ is the minimizing curve sought.

5. **Example.** If $F(z, z')$ satisfies (2.1) and is quasi-regular† (see (6.2)), every approach set A at every point z_0 consists of a single convex set; so (4.1) is trivially satisfied. Hence for such integrands there is always a minimizing

† Necessarily positive quasi-regular, for if it is negative quasi-regular and not linear, (2.1) cannot hold.

curve for $\mathcal{J}(C)$ in the class K of all curves joining two given points z_1 and z_2. This is, of course, a special case of a known theorem.

An example of an essentially different type is $\int F dt$ where

$$F(x, y, x', y') = (x'^2 + y'^2)^{1/2} + y^2[4(x'^2 + y'^2)^{1/2} - (x'^2 + 8y'^2)^{1/2}].$$

This integrand is not positive quasi-regular, for if $|y| > 1/2$, the $\mathcal{E}$-function can be negative. Given a set (x, y, p, q) with $p^2+q^2=1$, we seek to determine all $(\bar{p}, \bar{q})$ with $\bar{p}^2+\bar{q}^2=1$ such that

$$F_{x'}(x, y, \bar{p}, \bar{q}) = F_{x'}(x, y, p, q), \; F_{y'}(x, y, \bar{p}, \bar{q}) = F_{y'}(x, y, p, q). \tag{5.1}$$

We can write (5.1) in the form

$$\begin{aligned} p[1 + 4y^2 - y^2(8 - 7p^2)^{-1/2}] &= \bar{p}[1 + 4y^2 - y^2(8 - 7\bar{p}^2)^{-1/2}], \\ q[1 + 4y^2 - 8y^2(1 + 7q^2)^{-1/2}] &= \bar{q}[1 + 4y^2 - 8y^2(1 + 7\bar{q}^2)^{-1/2}]. \end{aligned} \tag{5.2}$$

The function in the left-hand member of the first of these equations is easily seen to have a positive derivative with respect to p for $|p| \leqq 1$; so the only solution of the first equation is $\bar{p}=p$. Since $p^2+q^2=\bar{p}^2+\bar{q}^2=1$, this implies $\bar{q}=\pm q$. Hence we substitute $-q$ for $\bar{q}$ in the second equation of (5.2). If it is not satisfied, then the entire approach set at (x, y) containing (p, q) is the set of multiples (kp, kq), $(k>0)$. If it is satisfied, then

$$1 + 4y^2 - 8y^2(1 + 7q^2)^{-1/2} = 0$$

or

$$q = \pm \left[\frac{1}{7}\left\{\left(\frac{8y^2}{1 + 4y^2}\right)^2 - 1\right\}\right]^{1/2}. \tag{5.3}$$

If q has the value (5.3), then A consists of the positive multiples of (p, q) and of $(p, -q)$, which are distinct if $q \neq 0$. Equation (5.3) can hold with $q \neq 0$ only for $|y| > 1/2$, and it is easy to see that $|q| < (3/7)^{1/2}$.

In any case, the approach set at (x, y) containing (p, q) consists either of a half-line or of two half-lines; and a half-line is a convex set. If the set consists of a single half-line, (4.1) clearly holds. Otherwise, let A consist of the multiples of (p, q) and of $(p, -q)$. If $y>0$, we define A_1 to be the multiples of $(p, -|q|)$ and A_2 to be the multiples of $(p, |q|)$; if $y<0$, we interchange the definitions of A_1 and A_2. Then each vector in A_1 has the form $(p_1, q_1) = (k_1 p, -k_1 \operatorname{sgn} y|q|)$, $(k_1>0)$, and each vector in A_2 has the form $(p_2, q_2) = (k_2 p, k_2 \operatorname{sgn} y|q|)$, $(k_2>0)$. Hence

$$\begin{aligned} \Omega_F(x, y; p_1, q_1; p_2, q_2) = 2y\{&q_1[4(p_2^2 + q_2^2)^{1/2} - (p_2^2 + 8q_2^2)^{1/2}] \\ &- q_2[4(p_1^2 + q_1^2)^{1/2} + (p_1^2 + 8q_1^2)^{1/2}]\}. \end{aligned}$$

The factors in square brackets are positive, and yq_1 and $-yq_2$ are negative (q_1 and q_2 are not zero, since by hypothesis A contains more than one unit vector). Hence the left member is negative, and (4.1) holds.

6. **Generalization to sets S with boundary points.** A slight generalization of Theorem 1 can be established at once. We need not assume that $z_2 \neq z_1$. The only use made of this hypothesis was to ensure that $|z_n'(t)|$ had a positive lower bound. If $z_2 = z_1$, we distinguish two cases.

Case I. The $|z_n'(t)|$ have a positive lower bound. In this case the preceding proof applies without change.

Case II. The lim inf $|z_n'(t)| = 0$. Since $|z_n'(t) = \mathcal{L}(\Pi_n)$ for almost all t, this implies that for a subsequence $\{\Pi_m\}$ of the minimizing sequence $\{\Pi_n\}$ we have $\mathcal{L}(\Pi_m) = 0$. Then $\mu = \lim \mathcal{J}(\Pi_n) = 0$; so the degenerate curve consisting of the single point z_1 is the curve sought.

Less trivial is the generalization which allows us to study integrands not defined for all z. Before stating this theorem it is desirable to introduce some definitions and establish a lemma.

(6.1) *The function $F(z, z')$ is positive (negative) regular the point z_0 if*† $u^\alpha F_{\alpha\beta}(z_0, p) u^\beta > 0$ (<0) *for all pairs u, p of orthogonal unit vectors.*

(6.2) *The function $F(z, z')$ is positive (negative) quasi-regular at z_0 if* $u^\alpha F_{\alpha\beta}(z_0, p) u^\beta \geqq 0$ $(\leqq 0)$ *for all pairs u, p of orthogonal unit vectors.*

We use the abbreviations p.q.r., n.q.r. for positive quasi-regular, negative quasi-regular, respectively. It is well known that $F(z, z')$ is p.q.r. (n.q.r.) at z_0 if and only if $\mathcal{E}(z_0, p, r) \geqq 0$ $(\leqq 0)$ for all $p \neq 0$ and all r. Although we do not use the concept in this note, we shall also make the following definition:

(6.3) *The function $F(z, z')$ is p.q.r. normal (n.q.r. normal) at z_0 if $\mathcal{E}(z_0, p, r)$ >0 (<0) whenever $p \neq 0$ and $r \neq kp$, $k \geqq 0$.*

With this terminology we state the following lemma:

LEMMA 5. *If $\phi_0'(t_0)$ and $\phi'(t_0)$ are defined, and $F(z, z')$ is p.q.r. at $z_0(t_0)$, then $\phi'(t_0) \geqq \phi_0'(t_0)$.*

Let ϵ be an arbitrary positive number, and let $F^\epsilon(z, z') = F(z, z') + \epsilon |z'|$. By an elementary computation, we find that if u and p are orthogonal unit vectors, then

$$(6.4) \qquad u^\alpha F^\epsilon_{\alpha\beta}(z, p) u^\beta = u^\alpha F_{\alpha\beta}(z, p) u^\beta + \epsilon.$$

The first term on the right is nonnegative at $z = z_0(t_0)$, by hypothesis. Hence the left-hand member is positive for all z of the set S in a neighborhood U

† $F_{ij}(z, z')$ means $F_{z^{i\prime} z^{j\prime}}(z, z')$.

of $z_0(t_0)$. If h is a sufficiently small positive number, less than† $1-t_0$, then for all large n the arcs $z=z_n(t)$ and $z=z_0(t)$, $(t_0 \leqq t \leqq t_0+h)$, are in U, and by a known semicontinuity theorem

$$\text{(6.5)} \qquad \liminf \int_{t_0}^{t_0+h} F^\epsilon(z_n, \dot{z}_n)dt \geqq \int_{t_0}^{t_0+h} F^\epsilon(z_0, \dot{z}_0)dt.$$

Recalling the definition of F^ϵ and the inequality $|\dot{z}_n| \leqq L$, we deduce that

$$\text{(6.6)} \qquad \liminf\, [\phi_n(t_0 + h) - \phi_n(t_0)] + \epsilon hL \geqq \phi_0(t_0 + h) - \phi_0(t_0).$$

That is,

$$\text{(6.7)} \qquad \phi(t_0 + h) - \phi(t_0) + \epsilon hL \geqq \phi_0(t_0 + h) - \phi_0(t_0).$$

If we divide by h and let h tend to 0, we obtain

$$\text{(6.8)} \qquad \phi'(t_0) + \epsilon L \geqq \phi_0'(t_0).$$

But ϵ is an arbitrary positive number; so (6.8) implies

$$\text{(6.9)} \qquad \phi'(t_0) \geqq \phi_0'(t_0),$$

which was to be proved.

Our next theorem is the following:

THEOREM 2. *Let S be a closed point set in q-dimensional space. If $F(z, z')$ satisfies condition* (2.1), *and condition* (4.1) *holds at every interior point of S, while $F(z, z')$ is positive quasi-regular at each boundary point of S, then in the class K there is a curve which minimizes $\mathcal{J}(C)$.*

Choose first a minimizing sequence C_n^* of curves $z=z_n^*(t)$, $(0 \leqq t \leqq 1)$. There is a set of values of t open relative to $[0, 1]$ for which $z_n^*(t)$ has distance greater than $1/n$ from the boundary of S. This open set consists of a finite or denumerable number of intervals, open relative to $[0, 1]$. Only a finite number h_n of these define arcs $C_{n,j}^*$, $(j=1, \cdots, h_n)$, of C_n^* of length greater than $1/n$. These arcs we replace by polygonal arcs $\Pi_{n,j}^*$ having the same end points and such that $|\mathcal{J}(\Pi_{n,j}^*)-\mathcal{J}(C_{n,j}^*)| < 1/nh_n$, $(j=1, \cdots, h_n)$. If we denote by C_n' the curve obtained by replacing the arcs $C_{n,j}^*$ by the polygonal arcs $\Pi_{n,j}^*$, then $|\mathcal{J}(C_n')-\mathcal{J}(C_n^*)| < 1/n$; so $\lim \mathcal{J}(C_n') = \mu$. By a minor modification of Lemma 1, each arc $\Pi_{n,j}^*$ can be replaced by an arc $\Pi_{n,j}$ having a number $s_{n,j}$ of vertices which is not greater than the number of vertices of $\Pi_{n,j}^*$ and which minimizes $\mathcal{J}(C)$ in the class of polygons joining the ends of $\Pi_{n,j}^*$ and having not more than $s_{n,j}$ vertices. Let C_n be the curve obtained by replacing the arcs $\Pi_{n,j}^*$ by the arcs $\Pi_{n,j}$. Then $\lim \mathcal{J}(C_n) = \mu$.

† If $t_0=1$, we take h negative; modifications are obvious.

Now let t_0 be a point of E. If $z_0(t_0)$ is interior to S, there is a neighborhood of $z_0(t_0)$ on which the C_n are polygonal if n is large. All the discussion leading up to equation (4.7) remains valid, for Lemmas 3 and 4 are purely local. So in this case (6.9) is valid. If $z_0(t_0)$ is a boundary point of S, then by Lemma 5 inequality (6.9) holds. Integrating, we obtain

$$\mu = \phi(1) \geqq \phi_0(1) = \mathcal{J}(C_0). \tag{6.10}$$

But C_0 is in K; so $\mathcal{J}(C_0) \geqq \mu$. This, with (6.10), proves $\mathcal{J}(C_0) = \mu$, and the theorem is established.

7. **Geometric interpretation of approach sets.** For fixed z, let us construct the graph in (u, p)-space of the function $u = F(z, p)$. As is well known, this is a conical hypersurface with vertex at the origin. Let A be an approach set at z. If p_1 and p_2 are both in A, then

$$F_i(z, p_1) = F_i(z, p_2), \qquad i = 1, \cdots, q. \tag{7.1}$$

But $u = p^\alpha F_\alpha(z, p_j)$ is the equation of the hyperplane tangent to $u = F(z, p)$ at p_j, $(j = 1, 2)$. Therefore equation (7.1) shows that this same hyperplane is tangent to the surface $u = F(z, p)$ at all points of A.

Conversely, let $u = l_\alpha p^\alpha$ be a hyperplane tangent to the hypersurface $u = F(z, p)$ for all p in a set A. Then if p is in A, the partial derivatives $F_i(z, p)$ must be the same as the partial derivatives l_i of the osculating function $l_\alpha p^\alpha$. Therefore $F_i(z, p) = l_i$ for all p in A, and A is an approach set at z.

If $F(z, p) > 0$ for $|p| > 0$, this interpretation can be formulated somewhat differently. In the preceding paragraphs set $u = 1$. The hyperplane $u = 1$ intersects $u = F(z, p)$ in a $(q-1)$-dimensional hypersurface, and the hyperplane $u = l_\alpha p^\alpha$ is tangent to $F(z, p) = 1$ if and only if $1 = l_\alpha p^\alpha$ is tangent to $F(z, p) = 1$. Thus if $q = 2$ and $F(z, p) > 0$ for $|p| > 0$, we can find all approach sets at z by constructing the curve $F(z, p) = 1$ and finding all the points (p^1, p^2) at which an arbitrary line $a_1 p^1 + a_2 p^2 + a_3 = 0$ is tangent to the curve.

This interpretation suggests that it might be more descriptive to replace the name "approach set" by "isotangential set."

8. **$\mathcal{E}$-admissibility of approach sets; a more general existence theorem.** The geometric interpretation in §7 suggests the following line of reasoning. If $z = z(t)$ minimizes $\mathcal{J}(C)$, then for each t_0 the surface $u = F(z(t_0), p)$ never sinks below the plane tangent to it at $p = z'(t_0)$. The analytical statement is $\mathcal{E}(z(t_0), z'(t_0), p) \geqq 0$ for all p. The definition of approach sets A was suggested by the Weierstrass-Erdman corner condition. Might it not be true that only those approach sets A are of importance which have the property that the hypersurface $u = F(z, p)$ never sinks below the hyperplane tangent to it at

the points p of A? We shall give this property of the set A the name "$\mathcal{E}$-admissibility." Stated analytically the definition is as follows:

(8.1) *If A is an approach set at z, it is $\mathcal{E}$-admissible if $\mathcal{E}(z, p_0, p) \geqq 0$ for all p_0 in A and all p.*

It would make no difference here if we replaced the words "all p_0 in A" by the words "some one p_0 in A." For if p_0, p_1 are any two vectors in A, then for all p the equation

$$\begin{aligned}\mathcal{E}(z, p_0, p) &= F(z, p) - p^\alpha F_\alpha(z, p_0) = F(z, p) - p^\alpha F_\alpha(z, p_1) \\ &= \mathcal{E}(z, p_1, p)\end{aligned} \tag{8.2}$$

holds.

The conjecture that approach sets which are not $\mathcal{E}$-admissable can be disregarded is indeed true, and we shall establish it in Theorem 3 below. However, the proof is rather complicated. It does not seem possible to rest upon the minimizing property of each Π_n, as we have done before. For in any direct analogue of the proof of the Weierstrass condition polygons are introduced which have new vertices, and Π_n does not necessarily minimize $\mathcal{J}(C)$ in the class of polygons with not more than s_n+1 vertices. The proof which we shall give is therefore based on the property of the sequence as a whole that $\lim \mathcal{J}(\Pi_n) = \mu$.

THEOREM 3. *Theorems* 1 *and* 2 *remain valid if in hypothesis* (4.1) *the words "each approach set" are replaced by "each $\mathcal{E}$-admissible approach set."*

In the proof of Theorems 1 and 2 the approach set A entered by way of Lemma 4, and then only in case there was no subsequence $\{\Pi_h\}$ and $\delta > 0$ for which either (a) $\lim_{h\to\infty}\theta(h, t_0-\delta, t_0+\delta) > t_0$ or (b) $\lim_{h\to\infty}\Theta(h, t_0-\delta, t_0+\delta) \leqq t_0$. For if either (a) or (b) held, the proof was relatively simple and did not involve any approach set A. We may suppose then that there is no subsequence $\{\Pi_h\}$ and no $\delta > 0$ for which either (a) or (b) holds. If we can then show that the approach set A in Lemma 4 is necessarily $\mathcal{E}$-admissible, our proof is complete.

In Lemma 4 there is no loss of generality in assuming that the common value of the partial derivatives $F_i(z_0(t_0), p)$ for all p in A is zero. For let us replace $F(z, p)$ by $F(z, p) - p^\alpha F_\alpha(z_0(t_0), p_0)$, where p_0 is in A. All the curves C of the family K join z_1 to z_2; so $\mathcal{J}(C)$ changes by $(z_2^\alpha - z_1^\alpha)F_\alpha(z_0(t_0), p_0)$, independent of C. Therefore the minimizing properties of the Π_n are unchanged by the alteration of $F(z, p)$. The statement that A is not $\mathcal{E}$-admissible assumes the form that there is a p_1 such that $F(z_0(t_0), p_1) < 0$. Because of the homogeneity of F, we may suppose that p_1 is a unit vector, and we write

$$F(z_0(t_0), p_1) = -7\gamma < 0.$$

By the continuity of F, there is an $\epsilon>0$ such that

$$F(z, p_1) < -6\gamma \tag{8.3}$$

if $|z-z_0(t_0)|<4\epsilon$. Furthermore, $F_i(z_0(t_0), p)=0$ if p is in A; hence we may suppose that ϵ has been chosen small enough so that the following statement holds:

(8.4) *The vector* $(F_1(z, p), \cdots, F_q(z, p))$ *has length less than* γ *if* $|z-z_0(t_0)|<4\epsilon$, $|z_2-z_1|\leqq|p|\leqq 2L+1$, *and* p *is in the* 3ϵ*-neighborhood of* A.

Clearly we may suppose $\epsilon<\min[1, |z_2-z_1|]$.

On the bounded closed set $|z-z_0(t_0)|\leqq 1$, $|p|\leqq 2L+1$, the functions $F_{z^i}(z, p)$ are continuous; hence we have the following statement:

(8.5) *There is an* $M>0$ *such that the vector* $(F_{z^i}(z, p), \cdots, F_{z^q}(z, p))$ *has length less than* M *if* $|z-z_0(t_0)|\leqq 1$ *and* $|p|\leqq 2L+1$.

Now from the subsequence $\{\Pi_m\}$ of Lemma 4 we choose a subsequence $\{\Pi_h\}$ with the following properties:

(8.6) $|z_h(t)-z_0(t)|<\epsilon$ for $0\leqq t\leqq 1$ and all h.

(8.7) The limits $t_1\equiv\lim_{h\to\infty}\theta(h, t_0-\delta, t_0+\delta)$ and $t_2\equiv\lim_{h\to\infty}\Theta(h, t_0-\delta, t_0+\delta)$ exist, where δ is prescribed by Lemma 4.

(As previously remarked, we need consider only the case $t_1\leqq t_0<t_2$.)

(8.8) δ is chosen less than ϵ/L.

(8.9) For all h the following inequality holds:

$$\Theta(h, t_0-\delta, t_0+\delta)-\theta(h, t_0-\delta, t_0+\delta)>(t_2-t_1)/2.$$

Each of the conditions (8.6), (8.7), (8.9) is readily satisfied by appropriate choice of a subsequence; for (8.6) and (8.9) we need only reject a finite number of terms.

Let $d_{h,1}, \cdots, d_{h,n(h)}$ be a sequence of points with the following properties:

(8.10) $d_{h,1}<d_{h,2}<\cdots$.

(8.11) $|d_{h,j+1}-d_{h,j}|<\gamma/M$ for $j=1, \cdots, n(h)-1$.

(8.12) $d_{h,1}=\theta(h, t_0-\delta, t_0+\delta)$, $d_{h,n(h)}=\Theta(h, t_0-\delta, t_0+\delta)$.

(8.13) Every t between $\theta(h, t_0-\delta, t_0+\delta)$ and $\Theta(h, t_0-\delta, t_0+\delta)$ which defines a vertex of Π_h is included among the $d_{h,j}$. (Thus $z_h(t)$ is linear on $d_{h,j}\leqq t\leqq d_{h,j+1}$.)

Let $m_{h,j}$ be the mid-point of the interval $[d_{h,j}, d_{h,j+1}]$.

On each interval $[d_{h,j}, d_{h,j+1}]$ we shall replace the line segment $z=z_h(t)$ belonging to Π_h by a polygon of two sides having the same beginning and the same end as the line segment. To keep the notation from becoming too complicated, we temporarily replace the symbols $d_{h,j}$, $m_{h,j}$, $d_{h,j+1}$, z_h by d_1, m, d_2, z, respectively. For $0 \leqq \tau \leqq \epsilon$ we define $\Pi(\tau)$ to be the two-sided polygon $z = z(t, \tau)$, where

$$(8.14)\qquad \begin{aligned} z(t, \tau) &= z(d_1) + (t - d_1)\tau p_1, & d_1 \leqq t < m, \\ z(t, \tau) &= z(2t - d_2) + (d_2 - t)\tau p_1, & m \leqq t \leqq d_2. \end{aligned}$$

We readily verify that $z(t, \tau)$ is continuous on $[d_1, d_2]$ and that

$$(8.15)\qquad z(d_1, \tau) \equiv z(d_1), \qquad z(d_2, \tau) \equiv z(d_2).$$

Also it is easy to show that

$$(8.16)\qquad \int_m^d F(z(t, 0), z'(t, 0))dt = \int_{d_1}^{d_2} F(z(t), z'(t))dt,$$

where the prime denotes the derivative with respect to t.

The polygon $\Pi(\tau)$ may be regarded as arising from $\Pi(0)$ by displacing the vertex $z(m, 0)$ by the amount $(d_2 - d_1)\tau p_1/2$. Hence we may calculate $\mathcal{J}'(\Pi(\tau))$ by applying (2.8) of I to the sides $[d_1, m]$, $[m, d_2]$ and adding. We obtain

$$(8.17)\qquad \begin{aligned} \frac{d}{d\tau}\mathcal{J}(\Pi(\tau)) = {} & \tfrac{1}{2}(d_2 - d_1)F_\alpha(z(\bar{t}, \tau), p_1)p_1^\alpha \\ & - \tfrac{1}{2}(d_2 - d_1)F_\alpha(z(\bar{\bar{t}}, \tau), z'(\bar{\bar{t}}, \tau))p_1^\alpha \\ & + \int_{d_1}^{d_2} F_{z^\alpha}(z(t, \tau), \dot{z}(t, \tau))p_1^\alpha \, | m - t | \, dt, \end{aligned}$$

where $d_1 < \bar{t} < m$ and $m < \bar{\bar{t}} < d_2$. Now we must prove the following statements:

(8.18) *If* $d_1 \leqq t \leqq d_2$ *and* $0 \leqq \tau \leqq \epsilon$, *then* $|z(t, \tau) - z_0(t_0)| < 3\epsilon$.

(8.19) *If* $m < t < d_2$ *and* $0 \leqq \tau \leqq \epsilon$, *then* $z'(t, \tau)$ *is in the* ϵ*-neighborhood of* A, *and* $|z_2 - z_1| \leqq |z'(t, \tau)| \leqq 2L + 1$.

We establish (8.19) first. By (8.14), we have

$$z'(t, \tau) = 2z'(2t - d_2) - \tau p_1.$$

But by our choice of parameter t, we have $|z_2 - z_1| \leqq z'(2t - d_2) \leqq L$, while $|\tau p_1| \leqq \epsilon < \min\,[|z_2 - z_1|, 1]$. Hence $|z_2 - z_1| < |z'(t, \tau)| < 2L + 1$. Also, by Lemma 4 the vector $z'(2t - d_2)$ is in the ϵ-neighborhood of A; whence $z'(t, \tau)$ is in the 3ϵ-neighborhood of A. Thus (8.19) is established.

By (c) of Lemma 2, $|\dot{z}(t)| \leqq L$. Hence if $t_0 - \delta \leqq t \leqq t_0 + \delta$, then (recalling (8.8)) we have

$$| z(t) - z(t_0) | \leqq L\delta < \epsilon.$$

By (8.6), $|z(t_0) - z_0(t_0)| < \epsilon$. Combining these inequalities, we find that both

$z(d_1)$ and $z(d_2)$ are in the 2ϵ-neighborhood of $z_0(t_0)$. The distance from $z(d_1) \equiv z(d_1, \tau)$ to $z(m, \tau)$ is $|(m-d_1)\tau p_1| < \epsilon$. So all three vertices $z(d_1, \tau)$, $z(m, \tau)$, and $z(d_2, \tau)$ are in the 3ϵ-neighborhood of $z_0(t_0)$, and (8.18) is proved.

Thus the arguments in (8.17) satisfy the requirements laid down in (8.3), (8.4), and (8.5), and we have from (8.17) and (8.11)

$$\begin{aligned}(8.20)\qquad \frac{d}{d\tau}\mathcal{J}(\Pi(\tau)) &\leqq -3\gamma(d_2-d_1) + \tfrac{1}{2}\gamma(d_2-d_1) + \int_{d_1}^{d_2} M\,|m-t|\,dt \\ &< -2\gamma(d_2-d_1) + M(d_2-d_1)^2 < -\gamma(d_2-d_1).\end{aligned}$$

If we integrate from $\tau=0$ to $\tau=\epsilon$, we obtain (using (8.16))

$$(8.21)\qquad \mathcal{J}(\Pi(\epsilon)) < \mathcal{J}(\Pi(0)) - \gamma\epsilon(d_2-d_1) = \mathcal{J}(\Pi) - \gamma\epsilon(d_2-d_1).$$

In the foregoing paragraphs $[d_1, d_2]$ was any one of the intervals $[d_{h,j}, d_{h,j+1}]$. Let the construction be carried out on each of these intervals. The arc $z=z_h(t)$, $d_{h,1}\leqq t\leqq d_{h,n(h)}$ is thereby replaced by a polygonal arc $z=z_h(t, \epsilon)$ having the same ends and satisfying the relation

$$\int_{h,1}^{d_{h,n(h)}} F(z_h(t,\epsilon), \dot z_h(t,\epsilon))dt < \int_{h,1}^{d_{h,n(h)}} F(z_h, \dot z_h)dt - \sum_{j=1}^{n(h)-1} \gamma\epsilon(d_{h,j+1} - d_{h,j}).$$

If we extend the range of definition of $z_h(t, \epsilon)$ by setting it equal to $z_h(t)$ for $0\leqq t<d_{h,1}$ and for $d_{h,n(h)}<t\leqq 1$, we obtain a polygon $\Pi_h(\epsilon)$ joining z_1 to z_2 and satisfying (by (8.21) and (8.9))

$$\begin{aligned}\mathcal{J}(\Pi_h(\epsilon)) < \mathcal{J}(\Pi_h) - \sum_{j=1}^{n(h)-1}\gamma\epsilon(d_{h,j+1}-d_{h,j}) &= \mathcal{J}(\Pi_h) - \gamma\epsilon(d_{h,n(h)} - d_{h,1}) \\ &< \mathcal{J}(\Pi_h) - \gamma\epsilon(t_2-t_1)/2.\end{aligned}$$

Therefore

$$\limsup_{h\to\infty} \mathcal{J}(\Pi_h(\epsilon)) \leqq \lim_{h\to\infty}\mathcal{J}(\Pi_h) - \gamma\epsilon(t_2-t_1)/2 = \mu - \tfrac{1}{2}\gamma\epsilon(t_2-t_1) < \mu.$$

This is impossible by the definition of μ, and our theorem is established.

It is evident that the restriction $z_2 \neq z_1$ can be removed here just as it was in §6. We thus arrive at the following theorem, which includes the three preceding ones as special cases:

THEOREM 4. *Let $F(z, z')$ be defined and satisfy the usual continuity and homogeneity conditions* for all z in a closed set S and all z'. Let $F(z, z')$ satisfy the following conditions*:

(i) *At every boundary point z of S, $F(z, z')$ is positive quasi-regular.*

(ii) *At every interior point z of S, every $\mathcal{E}$-admissible approach set A is*

* As specified in I.

the sum of a finite number of convex sets $A_1, \cdots, A_k$ *such that* $\Omega(z, p_i, p_j) < 0$ *if* p_i *is in* A_i, p_j *in* A_j, $i < j$.

Let condition (2.1) *be satisfied. Then for every two points* z_1, z_2 *of* S *the class* K *of all rectifiable curves lying in* S *and joining* z_2 *to* z_1 *either is empty or contains a minimizing curve for* $\mathcal{J}(C)$.

9. **Geometric interpretation of $\mathcal{E}$-admissibility.** The geometric interpretation of the property of $\mathcal{E}$-admissibility has already been given at the beginning of §8. If A is an approach set at z, it is $\mathcal{E}$-admissible if and only if the surface $u = F(z, p)$ never sinks below the hyperplane tangent to it at the points (u, p) with p in A. The alternative geometric interpretation of approach sets also carries with it an interpretation of $\mathcal{E}$-admissibility. Let the hyperplane (in p-space) $a_\alpha p^\alpha + b = 0$ be tangent to the hypersurface $F(z, p) = 1$ at all the points p of the surface which belong to A. This hyperplane divides p-space into two half-spaces. The half-space which contains the origin necessarily contains other points of $F(z, p) = 1$. The set A is then $\mathcal{E}$-admissible if and only if the entire hypersurface $F = 1$ lies in this same half-space.

It follows that if we form the least convex body Q containing $F(z, p) = 1$, then the points p of the hypersurface which are not on the boundary of Q cannot belong to any $\mathcal{E}$-admissible approach set. In particular, if $q = 2$, the points p_0 of $F(z, p) = 1$ on the boundary of Q either lie in a line segment belonging to the boundary of Q or they do not. In the second case the approach set A containing p_0 consists of the half-line from the origin through p_0. In the first case, let B be the set of all points p at which the line segment touches $F(z, p) = 1$. The (maximal) approach set at z containing p_0 then consists of all the half-lines from the origin through the points of B.

Consider, for example, the function $F(x, y, x', y')$ studied in §5. If $|y| \leqq 1/2$, the curve $F = 1$ is convex. If $|y| > 1/2$, the curve is dumbbell-shaped, curved inwards near its intersections with the x'-axis. It is then evident from the graph that there are just two approach sets which consist of more than one half-line, and these are determined by the points of contact with the two lines $x' = \text{const.}$ tangent to the curve at the points at which x' is respectively greatest and least. In §5 we showed rather more than this. We showed that these are the *only* approach sets other than half-lines; that besides these there are no other approach sets other than half-lines, whether $\mathcal{E}$-admissible or not. But Theorem 3 shows that the simpler conclusion reached here is sufficient to show the existence of minimizing curves for $\mathcal{J}(C)$.

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